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Resonances of the cusp family

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Abstract
We study a family of chaotic maps with limit cases—the tent map and the cusp map (the cusp family). We discuss the spectral properties of the corresponding Frobenius–Perron operator in different function spaces including spaces of analytical functions and study numerically the eigenvalues and eigenfunctions.

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1. Introduction

Resonances of dynamical systems are manifestations of the statistical properties of chaotic systems and describe the decay of correlations. Hence it is not surprising that they are studied very intensively. We refer the readers to recent reviews on this vast subject [1, 2]. Resonances appear also in the generalized spectra [3–5] of the evolution operators [6, 7] of chaotic maps.

The theory of resonances has been recently developed in terms of locally convex topological vector spaces [3–5]. This reflects the fact that dynamical systems are defined in terms of the space of observables and the evolution law. For different classes of observables the same evolution law may have different resonances, i.e. different rates of approach to equilibrium. However, once the class of observables is chosen the resonance structure is unique [5, 8]. Therefore we have proposed [5, 8, 9] that physical equivalence should reflect identical physical properties, i.e. rates of decay of correlations.

For many classes of maps, e.g. expanding maps, there exist some results about existence of resonances and their estimations [2]. However, for more complicated maps each case needs a separate consideration and results are sparse. Their study has attracted a lot of interest.

For example, the so-called cusp map [10] on the interval $[-1, 1]$

$$S(x) = 1 - 2\sqrt{|x|}$$

is an approximation of the Poincaré section of the Lorenz attractor [11, 12]. The absolutely continuous invariant probability measure of the cusp map has density

$$\rho(x) = \frac{1-x}{2}.$$

The cusp map is a limit case of the cusp family [13–15]:

$$S_\varepsilon\colon [-1,1] \to [-1,1] \qquad \varepsilon \in [0,1/2] \qquad \text{where} \quad S_\varepsilon(x) = \frac{1-\sqrt{1-4\varepsilon(1-\varepsilon-2|x|)}}{2\varepsilon}$$
$$\text{for} \quad \varepsilon \in (0,1/2] \qquad S_0(x) = \lim_{\varepsilon\downarrow 0} S_\varepsilon(x) = 1-2|x|. \tag{1}$$

The map with $\varepsilon = 0$ is the well-known tent map [16] while the map with $\varepsilon = 1/2$ is the cusp map [10].

The statistical analysis of dynamical systems is based on the Koopman and the Frobenius–Perron operators. Let Y be a set and $\mathcal{A}$ be a σ-algebra of measurable subsets of Y. The Koopman operator of a measurable map $S\colon Y \to Y$ acts on functions $f\colon Y \to \mathbb{C}$ as follows:

$$Vf(x) = f(Sx).$$

The Frobenius–Perron operator (FPO) U is defined with respect to a probability reference measure ν on $(Y, \mathcal{A})$. For $1 \leqslant p \leqslant \infty$ the FPO $U\colon L_p(Y,\mathcal{A},\nu) \to L_p(Y,\mathcal{A},\nu)$ is the dual of the operator $V\colon L_q(Y,\mathcal{A},\nu) \to L_q(Y,\mathcal{A},\nu)$, where $\frac{1}{p}+\frac{1}{q}=1$:

$$(U\rho|f) = (\rho|Vf) \qquad (\rho|f) = \int \nu(\mathrm{d}y)\overline{\rho(y)}f(y).$$

Usually one considers maps which possess an absolutely continuous invariant measure μ: $\mu(S^{-1}(\Delta)) = \mu(\Delta)$ for any measurable set $\Delta \in \mathcal{A}$.

A map S is called an exact endomorphism when any measurable set $\Delta \in \mathcal{A}$ of initial data with $\nu[\Delta] \neq 0$, eventually covers the whole space Y [6, 7]:

$$\lim_{n\to\infty} \nu[S^n\Delta] = \nu[Y]. \tag{2}$$

For an exact endomorphism, the absolutely continuous invariant measure μ is unique. Definition (2) is equivalent to the property that any probability density $\rho \in L_1(Y,\mathcal{A},\nu)$ approaches the equilibrium density ρ_{eq}:

$$\lim_{n\to\infty} \|U^n\rho - \rho_{\mathrm{eq}}\| = 0.$$

The FPO U is defined here with respect to the reference measure ν and $\rho_{\mathrm{eq}} = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}$. If the reference measure is the invariant measure μ then $\rho_{\mathrm{eq}} = 1$. In both cases 1 is an eigenvalue of the FPO:

$$U\rho_{\mathrm{eq}} = \rho_{\mathrm{eq}}.$$

In the case of dynamical systems on the interval $[\alpha, \beta]$, the reference measure is usually either the normalized Lebesgue measure or the invariant absolutely continuous probability measure. In our paper we use the invariant measure as the reference one.

The Frobenius–Perron operator U_ε of each member S_ε of the cusp family (1) with respect to the invariant measure μ_ε is

$$U_\varepsilon\rho(x) = \left(\tfrac{1}{2} - \varepsilon a_\varepsilon(x)\right)\rho(a_\varepsilon(x)) + \left(\tfrac{1}{2} + \varepsilon a_\varepsilon(x)\right)\rho(-a_\varepsilon(x)) \tag{3}$$

where

$$a_\varepsilon(x) = \frac{1-x}{2} - \frac{\varepsilon}{2}(1-x^2).$$

The absolutely continuous invariant probability measures μ_ε for the cusp family S_ε have densities [13]

$$\rho_\varepsilon(x) = \tfrac{1}{2} - \varepsilon x. \tag{4}$$

Each member S_ε of the cusp family is an exact system. For $\varepsilon \neq 1/2$ it follows directly from theorem 4 in section 8, ch 10 of [7], which gives sufficient conditions of exactness for piecewise monotonic maps. The exactness of the cusp map has been hinted by Hemmer [10] referring to the work of Lasota and Yorke [17]. The proposed hint seems to be irrelevant as the cusp map has a parabolic fixed point.

For the cusp map one should consider the so-called induced map [18–20] on the segment $[\sqrt{8} - 3, 3 - \sqrt{8}]$. This map satisfies the conditions of the above-mentioned theorem and therefore is exact. Since the exactness for a map and its induced map are equivalent [18–20], we obtain the exactness of the cusp map.

The objective of this paper is to study the resonances of the cusp family (1). In section 2 we present some definitions and results for the spectral theory of operators necessary for the study of the FPO of the cusp family. In section 3 we present results about the spectral properties of the FPO generated by the cusp family in different function spaces. In section 4 we analyse the spectral properties in spaces of analytic functions. In order to analyse the eigenvalues and eigenfunctions of the cusp family, we perform a numerical study in section 5. We show that the cusp family does not have a spectrum in the form r^n, where $n \in \mathbb{N}$, $r \in \mathbb{R}$, in the space of analytical functions, at least in the vicinity of the tent map. We analyse the behaviour of the eigenvalues in the vicinity of the cusp map. The behaviour of the eigenfunctions is also discussed.

2. Normal points of linear operators

Let A be a linear continuous operator in a locally convex topological linear space Φ. The point $z \in \mathbb{C}$ is said to be *regular* if the operator $A - zI$ has a continuous inverse, I is the identity operator. The set of all nonregular points is the spectrum of A, denoted as $\sigma(A)$. The point $z \in \mathbb{C}$ is said to be a *normal point* [21] if Φ admits a decomposition into a topological direct sum [22] of two closed linear subspaces

$$\Phi = \Phi_0 \oplus \Phi_1 \tag{5}$$

such that Φ_0 is finite dimensional, $A(\Phi_j) \subseteq \Phi_j$ for $j \in \{0, 1\}$, $(A - zI)\big|_{\Phi_1} : \Phi_1 \to \Phi_1$ has a continuous inverse and there exists $n \in \mathbb{N}$ such that $(A - zI)^n(\Phi_0) = \{0\}$.

Evidently the point z is regular if and only if it is normal and $\Phi_0 = \{0\}$. A normal point for which $\Phi_0 \neq \{0\}$ is called a *normal eigenvalue*.

It is well known [21, 23] that for any normal point z the decomposition (5) is unique. Moreover, the monotonic sequences of spaces $\ker(A - zI)^n$ and $(A - zI)^n(\Phi)$ stabilize and

$$\Phi_0 = \bigcup_{n=1}^{\infty} \ker(A - zI)^n \qquad \Phi_1 = \bigcap_{n=1}^{\infty} (A - zI)^n(\Phi). \tag{6}$$

For a normal point z we denote the eigenspace Φ_0 corresponding to the eigenvalue z as

$$\mathcal{E}(z, A) = \Phi_0. \tag{7}$$

Note that if z is regular then $\mathcal{E}(z, A) = \{0\}$. According to (6) the finite-dimensional space $\mathcal{E}(z, A)$ is spanned by the eigenvectors and the principal vectors of A associated with the eigenvalue z. In the case $\mathcal{E}(z, A) \neq \{0\}$ the dimension of $\mathcal{E}(z, A)$ is the *multiplicity* of the normal eigenvalue z.

If the spectrum of A is either finite or a sequence converging to 0 and any non-zero element of $\sigma(A)$ is a normal eigenvalue of A (this happens, e.g., for any compact operator on a Banach space [23]), we can renumber the spectrum $\sigma(A)$ as a sequence of eigenvalues of A,

$$z_n \qquad n = 0, 1, 2, \ldots$$

with the following properties:

$$\begin{aligned} &(1)\ |z_{n+1}| \leqslant |z_n| \text{ for all } n \in \mathbb{Z}_+ \\ &(2)\ \text{if } z_n \neq 0 \text{ then } z_n \in \sigma(A) \\ &(3)\ \text{if } z \in \sigma(A)\backslash\{0\} \text{ then } |\{n \in \mathbb{Z}_+ : z_n(A) = z\}| = \dim \mathcal{E}(z, A) \\ &(4)\ \text{if } |z_n| = |z_{n+1}| \text{ then } \arg z_n < \arg z_{n+1} \end{aligned} \tag{8}$$

where $\arg z \in (-\pi, \pi]$ is the argument of the complex number z.

3. Spectral properties of the Frobenius–Perron operator in L_p and C^k

Let us introduce the following notation:

$$\bar{D}(a, q) = \{z \in \mathbb{C} : |z - a| \leqslant q\} \qquad D(a, q) = \{z \in \mathbb{C} : |z - a| < q\}. \tag{9}$$

For any $p \in [1, +\infty]$ we denote the Hardy space in the disc $D(a, q)$ by $\mathcal{H}^p(a, q)$, i.e. $\mathcal{H}^p(a, q)$ is the space of holomorphic functions $f: D(a, q) \to \mathbb{C}$, which belong to $L^p(D(a, q))$ with respect to the Lebesgue measure. We endow this space with the L_p norm.

The operator $U_\varepsilon^X : X \to X$ is the restriction of U_ε to a locally convex function space X such that $U_\varepsilon(X) \subseteq X$. The spectrum of the operator U_ε^X is denoted by $\sigma\left(U_\varepsilon^X\right)$.

Proposition 1. *Let $\varepsilon \in [0, 1/2]$, X be either the Banach space $C[-1, 1]$ or $L_p([-1, 1], \mu_\varepsilon)$. Then the spectrum $\sigma\left(U_\varepsilon^X\right)$ coincides with the closed unit disc $\bar{D}(0, 1)$. Moreover, any z from the open unit disc $D(0, 1)$ is an eigenvalue of U_ε^X of infinite multiplicity. The point $z = 1$ is an eigenvalue of multiplicity 1.*

Proof. Since $\left\|U_\varepsilon^X\right\| = 1$ we have $\sigma\left(U_\varepsilon^X\right) \subseteq \bar{D}(0, 1)$. Let $z \in D(0, 1)$. Consider the Koopman operator of the cusp family $V_\varepsilon : X \to X$:

$$V_\varepsilon f(x) = f(S_\varepsilon(x)).$$

One can directly verify that the functions ψ,

$$\psi(x) = \sum_{k=0}^{\infty} z^k V_\varepsilon^k h \tag{10}$$

where $h(x) = g(x)(1 + 2\varepsilon x)$ and $g(x)$ is an odd function, are eigenfunctions of U_ε: $U_\varepsilon \psi = z\psi$. As g is an arbitrary odd function, this proves that all points of $D(0, 1)$ are eigenvalues of U_ε^X of infinite multiplicity. □

Remark 1. Formula (10) provides all the eigenfunctions of U_ε^X with eigenvalue z.

Remark 2. Proposition 1 and its proof remain valid for the Frobenius–Perron operator U of any continuous exact endomorphism (instead of h one should take any element of $\ker U$).

Proposition 2. *Let $\varepsilon \in [0, 1/2]$, $n = 1, 2, \ldots$, X be the Banach space $C^n[-1, 1]$. Then the spectrum $\sigma\left(U_\varepsilon^X\right)$ contains the closed disc $\bar{D}(0, (1/2 + \varepsilon)^{n+1})$, and any point of the open disc $D(0, (1/2 + \varepsilon)^{n+1})$ is a (non-normal) eigenvalue of U_ε^X of infinite multiplicity. The set $S = \sigma\left(U_\varepsilon^X\right)\backslash\bar{D}(0, (1/2 + \varepsilon)^n)$ is finite and any $z \in S$ is a normal eigenvalue of U_ε^X.*

Remark 3. Under the conditions of proposition 2 for any $z \in S$ and any $f \in \mathcal{E}(z, U_\varepsilon^X)$ the function f admits the analytic continuation to the disc $D(0, 1/\varepsilon - 1)$ if $\varepsilon \in (0, 1/2)$ and to the whole complex plane if $\varepsilon = 0$. This can be proved by estimating the growth of the sequence $s_n = \sup_{t\in[-1,1]} \left\| f^{(n)}(t) \right\|$.

Corollary 1. *Let $\varepsilon \in [0, 1/2)$, $X = C^\infty[-1, 1]$ with the natural topology* [22]. *Then the spectrum $\sigma(U_\varepsilon^X)$ is either finite or countable, $0 \in \sigma(U_\varepsilon^X)$ and any point $z \in \sigma(U_\varepsilon^X)\backslash\{0\}$ is a normal (and therefore isolated) eigenvalue of U_ε^X. Moreover for any $z \in \sigma(U_\varepsilon^X)\backslash\{0\}$ and any $f \in \mathcal{E}(z, U_\varepsilon^X)$ the function f admits the analytic continuation to the disc $D(0, 1/\varepsilon - 1)$ if $\varepsilon \in (0, 1/2)$ and to the whole complex plane if $\varepsilon = 0$.*

Corollary 2. *Let $n = 1, 2, \dots, \infty$ and X be the space $C^n[-1, 1]$. Then the spectrum $\sigma(U_{1/2}^X)$ is the closed unit disc $\bar{D}(0, 1)$, and any point of the open unit disc $D(0, 1)$ is an eigenvalue of $U_{1/2}^X$ of infinite multiplicity.*

Proof of proposition 2. Let us define the sequence t_n by the formula

$$t_0 = 1 \qquad t_{n+1} = -a_\varepsilon(t_n) \qquad n = 1, 2, \dots. \tag{11}$$

It is easy to see that $t_1 = 0$, the sequence t_n is strictly decreasing and

$$\begin{aligned} t_n &= -1 + \frac{4}{n} + O\left(\frac{1}{n^2}\right) && \text{for} \quad \varepsilon = 1/2, \\ t_n &= -1 + c(\varepsilon)\left(\frac{1}{2} + \varepsilon\right)^n + O\left(\left(\frac{1}{2} + \varepsilon\right)^{2n}\right) && \text{for} \quad \varepsilon \in [0, 1/2) \end{aligned} \tag{12}$$

where $c(\varepsilon) = 2 - 4\varepsilon + O(\varepsilon^2)$ is a constant.

Let $z \in \mathbb{C}$. Pick an arbitrary function $\phi\colon (0, 1] \to \mathbb{C}$. Define recurrently the function $f_\phi\colon (-1, 1] \to \mathbb{C}$ as follows

$$\begin{aligned} &f_\phi(x) = \phi(x) \text{ for } x \in (0, 1] = (t_1, t_0] \\ &f_\phi(x) = \frac{2zf\left(a_\varepsilon^{-1}(-x)\right)}{1 - 2\varepsilon x} - \frac{1 + 2\varepsilon x}{1 - 2\varepsilon x} f(-x) \qquad \text{for} \quad x \in (t_{n+1}, t_n] \quad n = 1, 2 \dots. \end{aligned} \tag{13}$$

It is straightforward to see that f_ϕ is the unique function $f\colon (-1, 1] \to \mathbb{C}$ for which $f|_{(0,1]} = \phi$ and $U_\varepsilon^X f(x) = zf(x)$ for all $x \in (-1, 1]$. Let now ϕ be an element of $C^\infty[0, 1]$ such that the support of ϕ (i.e. the closure in [0,1] of the set $\{t\colon \phi(t) \neq 0\}$) is contained in the interval $(0, -t_2)$. It is clear that $f_\phi \in C^\infty(-1, 1]$. Using formula (11) and the asymptotics (12), for any $z \in \mathbb{C}$, $|z| < (1/2 + \varepsilon)^{n+1}$ one can verify that

$$\lim_{t\downarrow -1} f_\phi^{(j)}(t) = 0 \qquad j = 0, 1, \dots, n. \tag{14}$$

Therefore, putting $f_\phi(-1) = 0$, we see that $f_\phi \in X = C^n[-1, 1]$ and $U_\varepsilon^X f_\phi = zf_\phi$. Hence $\sigma(U_\varepsilon^X)$ contains $\bar{D}(0, (1/2 + \varepsilon)^{n+1})$ and any point of $D(0, (1/2 + \varepsilon)^{n+1})$ is an eigenvalue of U_ε^X of infinite multiplicity. □

The second part of proposition 2 follows from Ruelle's results on spectra of positive transfer operators (see [2], theorem 2.5 and exercise 2.9).

4. Spectral properties of the operator U_ε in spaces of analytic functions

The spectral properties of the operator U_ε in spaces of analytical functions differ considerably depending on the choice of the space and on the values $\varepsilon = 1/2$ or $\varepsilon \neq 1/2$. Furthermore, not all of these properties are known yet.

Proposition 3. Let $\varepsilon \in (0, 1/2)$, $q \in (1, 1/\varepsilon - 1)$ and X be the Hardy space $\mathcal{H}^2(0, q)$. Then

(i) operator U_ε^X is nuclear
(ii) eigenvalues z_n of U_ε^X and the corresponding eigenspaces $\mathcal{E}(z_n, U_\varepsilon^X)$ do not depend on q
(iii) eigenvalues z_n satisfy the inequality

$$|z_n| \leqslant 1.5c^n \qquad \text{where} \quad c = c(\varepsilon) = \sqrt{1/2 + \sqrt{\varepsilon(1-\varepsilon)}} < 1. \tag{15}$$

We remind the reader [21] that a bounded operator A is nuclear or trace-class if tr$A$ is finite. An operator A is nuclear iff the absolute value operator $|A| = \sqrt{A^+A}$ is nuclear. If an operator A is compact, then the eigenvalues $s_n(A)$, $n = 0, 1, 2 \ldots$, of $|A|$ are known as the s-numbers of the operator A.

Proof of proposition 3. It is easy to show that for any $r > 1$

$$\alpha(r) = \sup_{|z|=r} |a_\varepsilon(z)| = \tfrac{1}{2}(1 - \varepsilon + r + \varepsilon r^2).$$

The function α is continuous and strictly increasing on the interval $(1, 1/\varepsilon - 1)$, and $\alpha(r) < r$ for any $r \in (1, 1/\varepsilon - 1)$. Put $q' = \alpha^{-1}(q) > q$. From the definition of the operator $U_\varepsilon(3)$, it follows that U_ε is a linear continuous operator from $\mathcal{H}^2(0, q)$ to $\mathcal{H}^2(0, q')$ with norm less than or equal to $1 + \varepsilon\alpha(q)$. Thus U_ε^X is the composition of U_ε and the operator J defining the embedding of $\mathcal{H}^2(0, q')$ into $\mathcal{H}^2(0, q)$. The s-numbers of the operator J are $s_n(J) = (q/q')^n$. Since $\sum_n s_n(J) < \infty$, the operator J is nuclear. As for any bounded operators A, B, $s_n(AB) \leqslant \|A\| s_n(B)$ [21], the operator U_ε^X is nuclear with s-numbers $s_n\left(U_\varepsilon^X\right) \leqslant (1 + \varepsilon\alpha(q))(q/q')^n$.

From remark 3 follows that the eigenvalues z_n of U_ε^X and the eigenspaces $\mathcal{E}(z_n, U_\varepsilon^X)$ do not depend on q and coincide with the eigenvalues and eigenspaces of $U_\varepsilon^{C^\infty[-1,1]}$.

From Weyl's inequality [24] we have

$$|z_n|^{n+1} \leqslant \prod_{k=0}^{n} |z_k| \leqslant \prod_{k=0}^{n} s_k\left(U_\varepsilon^X\right) \leqslant (1 + \varepsilon\alpha(q))^{n+1}(q/q')^{(n+1)n/2}$$

and therefore

$$z_n \leqslant (1 + \varepsilon\alpha(q))(q/q')^{n/2}. \tag{16}$$

The ratio q/q' is minimal for $q' = \sqrt{1/\varepsilon - 1}$ and is equal to $1/2 + \sqrt{\varepsilon(1-\varepsilon)}$. For this value of q' we have $1 + \varepsilon\alpha(q) \leqslant 1 + \sqrt{\varepsilon - \varepsilon^2} \leqslant 1.5$. Therefore inequality (16) for $q' = \sqrt{1/\varepsilon - 1}$ implies (15). □

The case $\varepsilon = 1/2$ is much more difficult and so far there exist very few results on the spectral properties of the Frobenius–Perron operators of the maps with parabolic neutral fixed points. We would like to point out the result of Rugh [25], who considered the Frobenius–Perron operators of piece-wise analytical maps, which are expanding everywhere except one parabolic fixed point. Namely, he constructed a specific map-dependent Banach space of analytical functions, where the spectrum of the FPO consists of the segment [0, 1] and some isolated normal eigenvalues. This space is in fact the image of $L_1[0, +\infty)$ with respect to some map-dependent integral transformation (similar to the Laplace transform). This idea applied to the cusp map allows us to verify that the FPO $U_{1/2}$ has similar spectral properties in certain weighted Hardy spaces in discs $D(\alpha, 1 + \alpha)$, $0 < \alpha < 1$.

The result of Rugh is very interesting since it provides the first example of a Banach space of smooth functions, where the spectrum of the Frobenius–Perron operator of the cusp map is non-trivial. Note that the functions of Rugh's space are analytic in all points of the segment except the parabolic fixed point (-1 in our case). However, we should note that the spectrum of the FPO of a map S in spaces of analytic functions with singularity at a fixed point of S may differ considerably from the spectrum in spaces of everywhere analytic functions. We illustrate this statement for the simplest expanding map F_0, which is the tent map.

Proposition 4. *Let $p \in [1, +\infty]$, $0 < \alpha < 1$ and $X = \mathcal{H}^p(\alpha, 1 + \alpha)$. Then the spectrum $\sigma\left(U_0^X\right)$ depends on p. Namely, $\sigma\left(U_0^X\right)$ is the union of the disc $\bar{D}\left(0, 2^{2/p-1}\right)$ and some set of (isolated) normal eigenvalues.*

Proof. Evidently, $U_0^X = A + B$, where

$$Af(x) = \frac{1}{2} f\left(\frac{1-x}{2}\right) \qquad Bf(x) = \frac{1}{2} f\left(\frac{x-1}{2}\right).$$

Since the image of the operator B is contained in the space $H^q(\alpha, \beta)$, where $\beta = \min\{1 + 5\alpha, 3 - \alpha\} > 1 + \alpha$, the operator B is nuclear and therefore compact.

Let us estimate now the norm of the operator A. Let $f \in X$. Then

$$\|Af\|^q = \int\limits_{D(\alpha,1+\alpha)} \left(\frac{1}{2}\left|f\left(\frac{x+\mathrm{i}y-1}{2}\right)\right|\right)^q \mathrm{d}x\,\mathrm{d}y = \int\limits_{D(\alpha-1,(1+\alpha)/2)} \frac{4}{2^q}|f(x+\mathrm{i}y)|^q\,\mathrm{d}x\,\mathrm{d}y$$

$$\leqslant \frac{1}{2^{q-2}} \int\limits_{D(\alpha,1+\alpha)} |f|^q\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2^{q-2}}\|f\|^q.$$

Therefore $\|A\| \leqslant 2^{\frac{2}{q}-1}$. On the other hand, one can verify that $Af_\lambda = 2^{-1-\lambda} f_\lambda$, where $f_\lambda(x) = (x+1)^\lambda$ and $f_\lambda \in X$ if and only if

$$\operatorname{Re}\lambda > -\frac{2}{q} \qquad \Longleftrightarrow \qquad \left|2^{-1-\lambda}\right| \leqslant 2^{\frac{2}{q}-1}.$$

Hence, the open disc $D\left(0, 2^{\frac{2}{q}-1}\right)$ is contained in the spectrum of A. Since $\|A\| \leqslant 2^{\frac{2}{q}-1}$, we find that $\sigma(A) = \bar{D}\left(0, 2^{\frac{2}{q}-1}\right)$.

Since the operator B is compact and $U_0^X = A + B$, the theorem on holomorphic operator-functions ([21], chapter I) implies that the spectrum of U_0^X is the union of $\bar{D}\left(0, 2^{\frac{2}{q}-1}\right)$ and some (isolated) normal eigenvalues. □

Proposition 5. *Let $0 < \nu < 0.3$ and X be the space of the functions $f\colon (-1, 1] \to \mathbb{C}$ such that the function $g_f(z) = f(-1 + 2^{-z})$, $g\colon [-1, +\infty) \to \mathbb{C}$ admits the analytic continuation to some element of the Hardy–Hilbert space H_R^2 in the half-plane $A_\nu = \{\operatorname{Re} z > -1 - \nu\}$ (We transfer the scalar product from the Hardy space H_R^2 to X by the bijective linear transform $f \mapsto g_f$). Then $\sigma\left(U_0^X\right) = [0, 1] \cup S$, where S consists of normal eigenvalues.*

Remark 4. The space X of proposition 5 is a Hilbert space of functions analytic on the set $D(-1, c)\backslash(-1 - c, -1]$ for some $c = c(\nu) > 2$.

Proof of proposition 5. From the definition of the scalar product in X, the operator $T\colon X \to H_R^2$, $Tf(x) = f(-1 + 2^{-x})$ is a unitary transformation. Therefore the operator

$$W = TU_0T^{-1}\colon H_R^2 \to H_R^2$$

and U_0 are unitarily equivalent. From the definitions of T and U_0 it follows that $W = A + B$, where

$$Af(x) = \tfrac{1}{2} f(x+1) \qquad Bf(x) = \tfrac{1}{2} f\left(-\log_2(2 + 2^{-y-1})\right).$$

It is straightforward to verify that the closure of the set $\left\{-\log_2(2+2^{-y-1})\colon y \in A_\nu\right\}$ is a compact subset of A_ν. Hence, the operator B is nuclear and therefore compact. On the other hand, the conventional Laplace transform and a linear change of variables provide a unitary equivalence between the operator A and the operator of multiplication with the function e^{-t} acting on a certain weighted Sobolev space of functions on $[0, +\infty)$. Therefore the spectrum of A is the segment $[0, 1]$.

Since the operator B is compact, the theorem on holomorphic operator-functions [21] implies that the spectrum of U_0^X, which is identical with the spectrum of W, is the union of the segment $[0, 1]$ and some set of (isolated) normal eigenvalues. □

It is worth noting that the space constructed in propositions 4, 5 is obtained by a method similar to the construction of Rugh [25]. Thus, it is not *a priori* clear what is the origin of the 'continuous spectrum' [0, 1] obtained in [25]: the dynamical properties of the map or the choice of the space.

In the space of real-analytical functions on $[-1, 1]$, the point spectrum of the Frobenius–Perron operator $U_{1/2}$ of the cusp map is $\{0, 1\}$ [26], i.e. the eigenfunction equation $U_{1/2} f = zf$ has non-zero analytic solutions only for $z = 0$ and $z = 1$. The proof of this result is technically complicated. Here, we prove a weaker result which admits a much simpler proof. Namely, we show that $\{0, 1\}$ is the point spectrum of $U_{1/2}$ in the space of entire functions.

Proposition 6. *Let $\varepsilon = 1/2$, X be the space of entire functions. Then the spectrum of U_ε^X is the whole complex plane $\mathbb{C}$ and the point spectrum of U_ε^X is the two-point set $\{0, 1\}$. The eigenvalue 0 has infinite multiplicity, and the eigenvalue 1 has multiplicity 1.*

Proof. The ergodicity of the map $S_{1/2}$ implies the multiplicity 1 for the eigenvalue $z = 1$. The null space of the operator $U_{1/2}$ is

$$\{f \in X\colon f(x) = (1+x)g(x)\colon g \text{ is an odd function}\}.$$

Therefore 0 is an eigenvalue of $U_{1/2}$ of infinite multiplicity. Let now $z \in \mathbb{C}\backslash\{0, 1\}$, $\psi \in X$ and $U_{1/2}\psi = z\psi$. The eigenvalue equation for $x = 1$ implies that $\psi(-1) = 0$. Therefore $\psi(x) = (1+x)g(x)$ for some $g \in X$. Let $\xi(x) = g(x) + g(-x)$. The eigenvalue equation $U_{1/2}\psi = z\psi$ in terms of the function ξ can be rewritten as

$$\xi\left(\left(\frac{x+1}{2}\right)^2\right) = \frac{32z\xi(x)}{x^3+5x^2+11x+15} + \frac{x^3-5x^2+11x-15}{x^3+5x^2+11x+15}\xi\left(\left(\frac{x-1}{2}\right)^2\right). \tag{17}$$

Let $M(R) = \max_{|x|=R} |\xi(x)|$, $c \in (0, \sqrt{2})$. It is easy to see that if $x \in \mathbb{C}$, $\operatorname{Re}(x+1)^2 \geqslant 0$, $\operatorname{Re} x \geqslant 0$, and $R \leqslant |(x+1)^2/4| \leqslant R + c\sqrt{R}$ then, for sufficiently large $R > 0$, $|x| \leqslant R$ and $|(x-1)^2/4| \leqslant R$. Since ξ is even this fact together with formula (17) imply that

$$M(R + \sqrt{R} + 1/4) \leqslant M(R)(1 + 5/\sqrt{R} + O(1/R)) \qquad \text{when} \quad R \to +\infty. \tag{18}$$

Applying (18) to $R_n = n^2/4$ and using the equality $R_{n+1} = R_n + \sqrt{R_n} + 1/4$, we obtain

$$M(n^2/4) \leqslant c_1 \prod_{k=1}^{n} (1 + 10/k + O(1/k^2))$$

for some positive constant c_1. Therefore $M(n^2/4) = O(n^{10})$, and $M(R) = O(R^5)$. This estimation implies (see [27]) that ξ is a polynomial of degree at most 5. On the other hand, using induction with respect to the degree of polynomial, one can show that there are no polynomial solutions of equation (17). □

Remark 5. A similar technique allows verification that for any $z \in \mathbb{C}$, $z \neq 0$ the function $f(x) = x$ does not belong to the space $U_{1/2}(X)$, where X is the space of entire functions. Therefore, the spectrum of $U_{1/2}^X$ is the whole complex plane.

5. Numerical results for the spectra

In the previous section we presented the general description of the spectrum of the operator U_ε. However, the eigenvalues and the eigenfunctions of the cusp family are not known explicitly. So we should compute them numerically. In order to perform this calculation in the space of the analytical functions, we use Taylor's expansion

$$f(x) = \sum_{k=0}^{N} c_k x^k. \tag{19}$$

The eigenvalue problem $U_\varepsilon f(x) = z f(x)$ can be reformulated in terms of the coefficients c_k:

$$U_\varepsilon f(x) = \sum_{k=0}^{N} c_k U_\varepsilon x^k = \sum_{k=0}^{N} c_k \sum_{p=0} a_{pk} x^p = z \sum_{k=0}^{N} c_k x^k. \tag{20}$$

As the operator U_ε is nuclear, we can project the last expression on the subspace $\{x_k\}_{k=0}^N$. Now the eigenvalue problem can be written as $A\vec{c} = z\vec{c}$, where $\{A\}_{kp} = a_{kp}$, see (20).

The coefficients a_{pk} in (20) are equal to

$$a_{pk} = \begin{cases} (-1)^p f(\varepsilon, k, p) & k \text{ is even} \\ (-1)^{p+1} 2\varepsilon f(\varepsilon, k+1, p) & k \text{ is odd} \end{cases} \tag{21}$$

where the function $f(\varepsilon, k, p)$ is defined as

$$f(\varepsilon, k, p) = \frac{1}{2^k} \sum_{l=0}^{p} \binom{k}{l} \binom{k}{p-l} \varepsilon^l (1-\varepsilon)^{k-l}$$

and $\binom{k}{l} = k!l!/(k-l)!$ is the binomial coefficient. The most precise and convenient way to calculate the coefficients a_{pk} is the use of the recurrence relation

$$a_{p,k+2} = \tfrac{1}{4}\left\{(1-\varepsilon)^2 a_{p,k} + (2\varepsilon - 2)a_{p-1,k} + (-2\varepsilon^2 + 2\varepsilon + 1)a_{p-2,k} - 2\varepsilon a_{p-3,k} + \varepsilon^2 a_{p-4,k}\right\}.$$

This representation is much more accurate than the numerical integration used in [14] hence it permits the use of longer expansion (19) without loss of accuracy.

It is worth noting that the matrix A is non-symmetric. Up to 2×10^3 terms in the expansion (19) were used to get converged results. In order to check convergence, we use the trace formula for the operator U_ε. Namely, as for $\varepsilon \in [0, 1/2)$ the operator U_ε is nuclear, we can calculate its trace by using the Grothendieck–Fredholm formula (see for example [2, 28]):

$$\operatorname{tr} U_\varepsilon = \sum_{n=0}^{\infty} z_n = \frac{1}{1/2 - \varepsilon} - \frac{2}{\sqrt{9 - 4\varepsilon(1-\varepsilon)}} \tag{22}$$

and compare this value with the numerical calculations.

In figure 1, ten maximal eigenvalues of the operator U_ε are presented. Because of the very good convergence of our numerical method for small ε, the asymptotics of the z_n as $\varepsilon \to 0$ can be numerically calculated:

$$\frac{z_{n+1}}{z_n} = \frac{1}{4} + \left(2n - \frac{1}{2}\right)\varepsilon + O(\varepsilon^2). \tag{23}$$

Hence the cusp family has neither spectrum in the form r^n, where $n \in \mathbb{N}$, $r \in \mathbb{R}$, nor combination of a few such spectra when $\varepsilon \neq 0$.

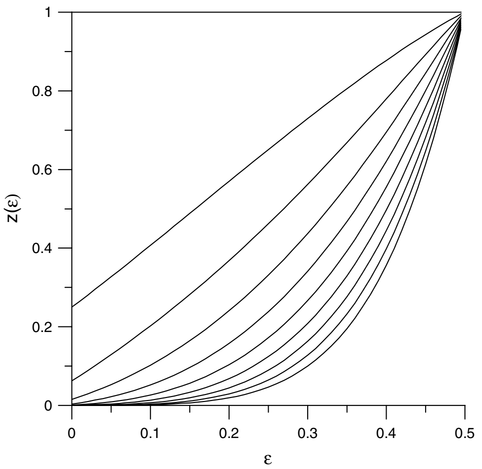

Figure 1. Ten maximal eigenvalues as functions of ε.

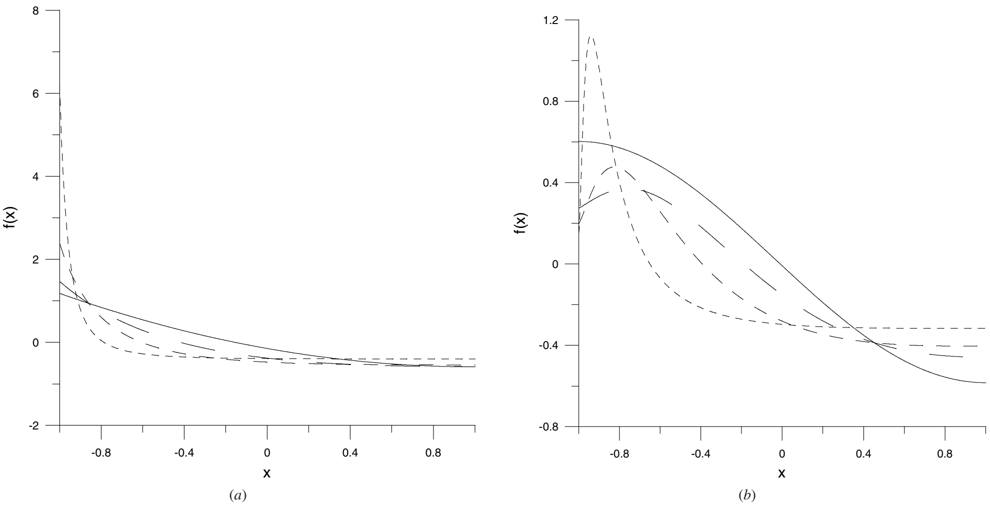

Figure 2. The second (*a*) and fourth (*b*) eigenfunctions for $\varepsilon = 0$ (the solid line), 0.25 (the long-dashed line), 0.4 (the dashed line) and 0.48 (the short-dashed line).

Using relation (23), we can find a general formula for the eigenvalues when ε is small:

$$z_{n+1} = \left(\tfrac{1}{4}\right)^n (1 + 2n(2n+1)\varepsilon + O(\varepsilon^2)) \qquad n = 0, 1, 2, \ldots . \tag{24}$$

This result gives for the asymptotics of the trace

$$\operatorname{tr} U_\varepsilon = \frac{4}{3} + \frac{104}{27}\varepsilon + O(\varepsilon^2) \qquad \text{when} \quad \varepsilon \to 0. \tag{25}$$

Formula (25) coincides with the asymptotics of equation (22). This coincidence supports strongly formulae (23), (24) which are obtained only numerically.

When $\varepsilon \to 1/2$ and n is fixed, one can see that $z_n \to 1$. This result agrees with the divergence of the trace. We have also checked that the eigenvalues have the asymptotics

$$z_n = (1/2 + \varepsilon)^n \qquad \text{when} \quad \varepsilon \to 1/2 \tag{26}$$

that agrees with the asymptotics found in [14].

Let us now discuss the eigenfunction behaviour. In figures 2(*a*) and (*b*) we present the second and fourth eigenfunctions, respectively, for few values of ε. One can easily see a concentration effect in the vicinity of -1 as $\varepsilon \to 1/2$. The eigenfunctions tend to have the support only at the point $x = -1$. This behaviour is in good agreement with the existence of a 'formal eigenfunction' $\delta(x + 1)$ for $\varepsilon = 0.5$. Such behaviour of eigenfunctions supports numerically the non-existence of a non-trivial (except for $\{0, 1\}$) spectrum for the cusp map in the space of the real analytic functions [26] as the limit functions have a singularity at the point -1.

6. Conclusions

The spectral properties of the cusp family (1) that 'interpolates' between the tent map and the cusp map have been investigated in different function spaces. While some results (propositions 1, 2 and 6) about the spectrum of the cusp map have been proved, the general description for different spaces of analytic functions is still unknown.

There are a few questions which are particularly interesting in this context. First, the question about the asymptotics of the autocorrelation function for the cusp map. As the resonance eigenvalues tend to unity, one can expect non-exponential decrease of the autocorrelation function. The estimations in paper [15] show that the autocorrelation function $C(n)$ decreases as $1/n$ when $n \to \infty$. However, this conjecture is not yet analytically proved. Another question addresses the choice of the space of analytic functions where the spectrum of the FPO is naturally defined by the dynamics of the map. Moreover, our calculations and the calculations of [14] show that the spectrum of the cusp family is real. While there are some analytical results about the reality of the spectrum [29], they are not applicable to the cusp family. Hence the question about the reality of the spectrum also remains open.

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